ON STABILITY OF SAMPLING-RECONSTRUCTION MODELS

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Abstract. A useful sampling-reconstruction model should be stable with respect to different kind of small perturbations, regardless whether they result from jitter, measurement errors, or simply from a small change in the model assumptions. In this paper we prove this result for a large class of sampling models. We define different classes of perturbations and quantify the robustness of a model with respect to them. We also use the theory of localized frames to study the frame algorithm for recovering the original signal from its samples.

1. Introduction

The sampling and reconstruction problem includes devising efficient methods for representing a signal (function) in terms of a discrete (finite or countable) set of its samples (values) and reconstructing the original signal from the samples (see e.g., [1, 3, 8, 9, 17, 22] and the reference therein). In this paper we consider a very general sampling model where the signal is assumed to belong to a finitely generated shift invariant space and the sampling is performed on an irregular separated set and is averaged by finite Borel measures. The main focus of this paper is on describing and quantifying "admissible" perturbations of the sampling model which may result from altering the sampling set (jitter) (see e.g. [6, 7, 14]), or the averaging sampling measures (measuring devices) or the generators of the underlying shift-invariant space (see e.g., [5, 18]).

As recently became customary in sampling theory (see e.g. [1, 3, 11, 19, 20, 21, 22]), we mesh operator theory techniques and those of shift invariant and Wiener amalgam spaces [13]. The latter provide us with relatively straight-forward proofs while the former allow us to keep in sight our objective. In section 2 we show that all the properties of our sampling model can be encoded in the sampling operator U.

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The sampling model admits reconstruction if its sampling operator is bounded both above and below. Our first goal is to show that any and all of the small perturbations mentioned above result in a small perturbation of U in the operator norm. This will prove the stability of sampling in our model with respect to those perturbations and the corresponding estimates we obtain will quantify this stability. Our second goal is to show how a frame algorithm can be used to reconstruct signals in our sampling model. Finally, our last goal is to show that the reconstruction error due to the perturbations we describe is controlled continuously by the perturbation errors.

The paper is organized as follows. In section 2 we describe our sampling model, introduce relevant notions and notation, and cite a few preliminary results. The main results are presented in section 3. Perturbation results addressing our first goal are in subsection 3.1. There we prove that a set of sampling remains such under a small perturbation of the sampling measures and/or the generators of the shift invariant space. It is also shown that sampling remains stable with respect to a perturbation of the sampling set itself. In subsection 3.2 we show that, in case of a signal in a Hilbert space, a frame algorithm can be used to reconstruct the function from its samples. We also use the results of the previous subsection and the theory of localized frames to show that under mild additional assumptions a set of sampling for a Hilbert shift invariant space is also a set of sampling for a chain of Banach shift invariant spaces to which the frame algorithm extends. In subsection 3.3 we study the dependence of the reconstruction error upon the perturbation errors. The proofs of the results in section 3 are relegated to section 4.

2. Description of the sampling model

This section is primarily devoted to introduction of the sampling model we use in this paper. We also present most of the necessary notation and cite some of the preliminary results that will be used later.

The signals we are studying in this paper are represented by functions $f \in L^p(\mathbb{R}^d)$, for some $p \in [1, \infty]$ and $d \in \mathbb{N}$. Moreover, we assume that f belongs to a shift invariant space

$$V^p(\Phi) = \{\sum_{k\in\mathbb{Z}^d} C_k^T \Phi_k : C \in (\ell^p(\mathbb{Z}^d))^{(r)}\}. \tag{2.1}$$

Here $\Phi = (\phi^1, \dots, \phi^r)^T$ is a vector of functions, $\Phi_k = \Phi(\cdot - k)$, and $C = (c^1, \dots, c^r)^T$ is a vector of sequences belonging to $(\ell^p(\mathbb{Z}^d))^{(r)}$. Among

the equivalent norms in $(\ell^p(\mathbb{Z}^d))^{(r)}$ we choose

$$\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} = \sum_{i=1}^{r} \|c^i\|_{\ell^p(\mathbb{Z}^d)}.$$

In order to avoid convergence issues in (2.1) we assume that the set $\{\phi^1(\cdot - k), \dots, \phi^r(\cdot - k); k \in \mathbb{Z}^d\}$ generates an unconditional basis for $V^p(\Phi)$. In particular, we require that there exist constants $0 < m_p \leq M_p < \infty$, such that

(2.2)
$$m_p\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \leq \|\sum_{k\in\mathbb{Z}^d} C_k^T \Phi_k\|_{L^p} \leq M_p\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}},\ \forall C \in (\ell^p(\mathbb{Z}^d))^{(r)}.$$

The unconditional basis assumption (2.2) implies [3] that the space $V^p(\Phi)$ is a closed subspace of $L^p(\mathbb{R}^d)$.

Since we are interested in sampling in $V^p(\Phi)$ we add an assumption that would make all the functions in these spaces continuous and, therefore, pointwise evaluations will be meaningful. To this end, we assume that all generators Φ belong to a Wiener-amalgam space $(W_0^1)^{(r)}$ as defined below. For $1 \leq p < \infty$, a measurable function f belongs to W^p if it satisfies

$$\|f\|_{W^p} = \left(\sum_{k\in\mathbb{Z}^d} \operatorname*{esssup}_{x\in[0,1]^d} |f(x+k)|^p\right)^{1/p} < \infty. \tag{2.3}$$

If $p = \infty$, a measurable function f belongs to W^∞ if it satisfies

$$\|f\|_{W^\infty} = \sup_{k\in\mathbb{Z}^d}\{\operatorname*{esssup}_{x\in[0,1]^d} |f(x+k)|\} < \infty. \tag{2.4}$$

Hence, W^∞ coincides with $L^\infty(\mathbb{R}^d)$. It is well known that W^p are Banach spaces [13], and clearly $W^p \subseteq L^p$. By $(W^p)^{(r)}$ we denote the space of vectors $\Psi = (\psi^1, \dots, \psi^r)^T$ of W^p-functions with the norm

$$\|\Psi\|_{(W^p)^{(r)}} = \sum_{i=1}^{r} \|\psi^i\|_{W^p}.$$

The closed subspace of (vectors of) continuous functions in W^p (respectively, $(W^p)^{(r)}$) will be denoted by W_0^p (or $(W_0^p)^{(r)}$).

In this paper we are interested in average sampling performed by a vector of measures. We denote by $\mathcal{M}(\mathbb{R}^d) = \mathcal{M}_0(\mathbb{R}^d)$ the Banach space of finite complex Borel measures on $\mathbb{R}^d$. The norm on $\mathcal{M}(\mathbb{R}^d)$ is given by $\|\mu\| = \int_{\mathbb{R}^d} d|\mu|(y)$, i.e., the total variation of a measure μ. By $(\mathcal{M}(\mathbb{R}^d))^{(t)}$ we denote the space of vectors $\overrightarrow{\mu} = (\mu^1, \dots, \mu^t)$ of measures from $\mathcal{M}(\mathbb{R}^d)$ with the norm $\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} = \sum_{j=1}^{t} \|\mu^j\|$. The symbols

$\mathcal{M}_s(\mathbb{R}^d)$ $((\mathcal{M}_s(\mathbb{R}^d))^{(t)})$, $0 \le s < \infty$, will be used for the subspace of $\mathcal{M}(\mathbb{R}^d)$ $((\mathcal{M}(\mathbb{R}^d))^{(t)})$ of all (vectors of) measures $\mu \in \mathcal{M}(\mathbb{R}^d)$ such that $(1+|x|)^s \in L^1(\mathbb{R}^d, d|\mu|)$, i.e., $\int (1+|x|)^s d|\mu|(x) < \infty$. By $\mathcal{M}_\infty(\mathbb{R}^d)$ $((\mathcal{M}_\infty(\mathbb{R}^d))^{(t)})$ we denote the space of all (vectors of) measures with compact support. Clearly $\mathcal{M}_s(\mathbb{R}^d) \subset \mathcal{M}_r(\mathbb{R}^d)$ for $0 \le r \le s \le \infty$.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$ and a measurable function ϕ on $\mathbb{R}^d$, the convolution of the function ϕ and the measure μ is defined by

$$(\phi * \mu)(x) = \int_{\mathbb{R}^d} \phi(x-y) d\mu(y), \quad x \in \mathbb{R}^d.$$

When we have a vector of measurable functions $\Phi = (\phi^1, \dots, \phi^r)^T$ and a vector of finite complex Borel measures $\overrightarrow{\mu} = (\mu^1, \dots, \mu^t)$, then the convolution $\Phi * \overrightarrow{\mu}$ is the $r \times t$ matrix given by

$$\Phi * \overrightarrow{\mu} = \begin{pmatrix} \phi^1 * \mu^1 & \dots & \phi^1 * \mu^t \\ \vdots & & \vdots \\ \phi^r * \mu^1 & \dots & \phi^r * \mu^t \end{pmatrix}.$$

Let J be a countable index set and $X = \{x_j : j \in J\}$ be a subset of $\mathbb{R}^d$. The reconstruction problem in our sampling model consists of finding the function $f \in V^p(\Phi)$ from the knowledge of its samples

$$(f * \overrightarrow{\mu})(X) = \{(f * \overrightarrow{\mu})(x_j) = \big((f * \mu^1)(x_j), \dots, (f * \mu^t)(x_j)\big)\}_{j \in J}.$$

When $t = 1$ and $\mu = \delta_0$, i.e., μ is the Dirac measure on $\mathbb{R}^d$ concentrated at zero, then $(f * \overrightarrow{\mu})(X) = \{f(x_j)\}_{j \in J}$ and we obtain the classical (ideal) sampling model. When $d\overrightarrow{\mu} = \Psi dx$, where $\Psi \in (L^1(\mathbb{R}^d))^{(t)}$ and dx is the Lebesgue measure on $\mathbb{R}^d$, i.e., $\overrightarrow{\mu}$ is absolutely continuous with respect to the Lebesgue measure, then we write $(f * \Psi)(X)$ instead of $(f * \overrightarrow{\mu})(X)$, and our model is reduced to the case analyzed in [5].

Definition 2.1. Let $1 \le p \le \infty$ and $X = \{x_j : j \in J\}$ be a countable subset of $\mathbb{R}^d$. We say that X is a *set of sampling* for $V^p(\Phi)$ and $\overrightarrow{\mu}$ (or, simply, a $\overrightarrow{\mu}$-sampling set for $V^p(\Phi)$) if there exist constants $0 < A_p \le B_p < \infty$ such that

$$A_p \|f\|_{L^p} \le \|(f * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \le B_p \|f\|_{L^p}, \text{ for all } f \in V^p(\Phi). \tag{2.5}$$

If $d\overrightarrow{\mu} = \Psi dx$ then a $\overrightarrow{\mu}$-sampling set X will be called a Ψ-sampling set and, if $t = 1$ and $\mu = \delta_0$, then X will be called an *ideal* sampling set. To ensure that an upper bound B_p in (2.5) always exists (see (4.2)) we restrict our attention only to separated sets X.

Definition 2.2. We say that X is *separated* if there exists $\delta > 0$ such that $\inf_{i,j \in J, i \ne j} |x_i - x_j| \ge \delta$. The number δ is called the *separation constant* of the set X.

It is not hard to extend our results to the case of a finite union of separated sets. We do not, however, pursue this relatively trivial but space consuming generalization.

Definition 2.3. Let $\overrightarrow{\mu} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, $\Phi \in (W_0^1)^{(r)}$ satisfy (2.2), and $X = \{x_j, j \in J\} \subset \mathbb{R}^d$ be a separated set. The *sampling model* is the triple $(X, \Phi, \overrightarrow{\mu})$. The sampling model $(X, \Phi, \overrightarrow{\mu})$ is called *p-stable* if X is a $\overrightarrow{\mu}$-sampling set for $V^p(\Phi)$, $p \in [1, \infty]$.

Given a sampling model $(X, \Phi, \overrightarrow{\mu})$ we proceed to define its sampling operator.

Definition 2.4. The *sampling operator* $U = U_{(X,\Phi,\overrightarrow{\mu})} : (\ell^p(\mathbb{Z}^d))^{(r)} \to (\ell^p(J))^{(t)}$ is defined by $UC = (f * \overrightarrow{\mu})(X)$, where $f = \sum\limits_{k\in\mathbb{Z}^d} C_k^T \Phi_k \in V^p(\Phi)$.

We can think of U as a $t \times r$ matrix of operators

$$U = \begin{pmatrix} U^{1,1} & \dots & U^{r,1} \\ \vdots & & \vdots \\ U^{1,t} & \dots & U^{r,t} \end{pmatrix},$$

where for each $1 \le i \le r$ and $1 \le l \le t$ the operator $U^{i,l}$ is defined by an infinite matrix with entries $(U^{i,l})_{j,k} = (\phi^i * \mu^l)(x_j - k)$, $j \in J$, $k \in \mathbb{Z}^d$. The operator norm of U is given by $\|U\|_{p,op} = \sum_{l=1}^t \sum_{i=1}^r \|U^{i,l}\|$.

The following proposition shows that all the interesting properties of a sampling model $(X, \Phi, \overrightarrow{\mu})$ are, indeed, encoded in the sampling operator U. The proof of this result follows immediately from (2.2) and (2.5).

Proposition 2.1. *The sampling model $(X, \Phi, \overrightarrow{\mu})$ is p-stable if and only if there exist $0 < \eta_p \le \beta_p < \infty$ such that for all $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$ the sampling operator U satisfies*

$$\eta_p \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \le \|UC\|_{(\ell^p(J))^{(t)}} \le \beta_p \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}. \tag{2.6}$$

The next lemma is, essentially, a nutshell for many of the results in this paper.

Lemma 2.2. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model and U be its sampling operator satisfying (2.6). Let also $(\widetilde{X}, \Theta, \overrightarrow{\alpha})$ be a sampling model such that its sampling operator U_Δ satisfies $\|U - U_\Delta\| < \eta_p$. Then $(\widetilde{X}, \Theta, \overrightarrow{\alpha})$ is also p-stable.*

Proof. Let $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$. Then

$$\begin{aligned} \|U_\Delta C\|_{(\ell^p(J))^{(t)}} &\le \|(U - U_\Delta)C\|_{(\ell^p(J))^{(t)}} + \|UC\|_{(\ell^p(J))^{(t)}} \\ &\le \|U - U_\Delta\| \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} + \beta_p \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}. \end{aligned}$$

Therefore, since $\|U - U_\Delta\| < \eta_p$, then we have

$$\|U_\Delta C\|_{(\ell^p(J))^{(t)}} \leq (\eta_p + \beta_p) \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}. \tag{2.7}$$

On the other hand, since

$$\begin{aligned} \eta_p \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} &\leq \|UC\|_{(\ell^p(J))^{(t)}} \leq \|(U - U_\Delta)C\|_{(\ell^p(J))^{(t)}} + \|U_\Delta C\|_{(\ell^p(J))^{(t)}} \\ &\leq \|U - U_\Delta\| \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} + \|U_\Delta C\|_{(\ell^p(J))^{(t)}}. \end{aligned}$$

Hence,

$$(\eta_p - \|U - U_\Delta\|) \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \leq \|U_\Delta C\|_{(\ell^p(J))^{(t)}}. \tag{2.8}$$

Since $\|U - U_\Delta\| < \eta_p$, the conclusion of the lemma follows from (2.7), (2.8), and Proposition 2.1. □

3. Main Results

In this section we collect the main results of our paper.

3.1. Admissible perturbations of a sampling model.

In practice, shift invariant spaces are used to model classes of signals that can occur (or that are allowed) in applications. However often, the functions in a shift invariant space model only give approximations to the signals of interest. For this reason, we begin with a result where the perturbation of a sampling model is due to a small change of the genetators of the underlying shift invariant space.

Theorem 3.1. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$. Then there exists $\epsilon_0 > 0$ such that the sampling model $(X, \Theta, \overrightarrow{\mu})$ is also p-stable, whenever $\Theta \in (W_0^1)^{(r)}$ and $\|\Phi - \Theta\|_{(W^1)^{(r)}} < \epsilon_0$.*

The above result means that if $\overrightarrow{\mu} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, $\Phi \in (W_0^1)^{(r)}$ satisfies (2.2), $X = \{x_j, j \in J\} \subset \mathbb{R}^d$ is a separated $\overrightarrow{\mu}$-sampling set for $V^p(\Phi)$, and Θ satisfies the assumptions of the theorem, then there exist $0 < A_p^{'} \leq B_p^{'} < \infty$ such that

$$A_p^{'} \|g\|_{L^p} \leq \|(g * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \leq B_p^{'} \|g\|_{L^p}, \text{ for all } g \in V^p(\Theta). \tag{3.1}$$

In the proof of this result in section 4 we will provide explicit estimates for ϵ_0 and the bounds $A_p^{'}$ and $B_p^{'}$.

As a consequence of Theorem 3.1 we have the following results that were first proved in [5]. The proofs now are immediate: we apply Theorem 3.1 with $d\overrightarrow{\mu} = \Psi dx$ for Corollary 3.2 and $\overrightarrow{\mu} = \delta_0$ for Corollary 3.3.

Corollary 3.2. *Let $\Psi \in (L^1(\mathbb{R}^d))^{(t)}$, $\Phi \in (W_0^1)^{(r)}$ satisfy (2.2), and $X = \{x_j, j \in J\} \subset \mathbb{R}^d$ be a separated Ψ-sampling set for $V^p(\Phi)$. Then there exists $\epsilon_0 > 0$ such that X is a Ψ-sampling set for $V^p(\Theta)$, whenever $\Theta \in (W_0^1)^{(r)}$ and $\|\Phi - \Theta\|_{(W^1)^{(r)}} < \epsilon_0$.*

Corollary 3.3. *Let $\Phi \in (W_0^1)^{(r)}$ satisfying (2.2) and $X = \{x_j, j \in J\} \subset \mathbb{R}^d$ be a separated ideal set of sampling for $V^p(\Phi)$. Then there exists $\epsilon_0 > 0$ such that X is an ideal set of sampling for $V^p(\Theta)$, whenever $\Theta \in (W_0^1)^{(r)}$ and $\|\Phi - \Theta\|_{(W^1)^{(r)}} \le \epsilon < \epsilon_0$.*

In practice, signal samples are obtained using measuring devices with characteristics that are not fully known, and the measurements reflect local averages rather than exact sample values. Thus, a sampling measure $\overrightarrow{\mu}$ is a model that approximate the characterisitics of a measuring device. For this reason, the next theorem describes the case when the perturbation is due to some uncertainty about the characteristics of the measuring devices, that is a perturbation of the vector of measures $\overrightarrow{\mu}$.

Theorem 3.4. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$. Then there exists $\epsilon_0 > 0$ such that the sampling model $(X, \Phi, \overrightarrow{\alpha})$ is also p-stable, whenever $\overrightarrow{\alpha} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$ and*

$$\|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0.$$

Again, if X, $\overrightarrow{\mu}$, $\overrightarrow{\alpha}$, and Φ satisfy the assumptions of the theorem then there exist $0 < A_p^{'} \le B_p^{'} < \infty$ such that

$$A_p^{'}\|f\|_{L^p} \le \|(f * \overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}} \le B_p^{'}\|f\|_{L^p}, \text{ for all } f \in V^p(\Phi), \tag{3.2}$$

and the explicit estimates for ϵ_0, $A_p^{'}$ and $B_p^{'}$ will be given in section 4.

Considering $\overrightarrow{\mu}$ and $\overrightarrow{\alpha}$ in Theorem 3.4 such that $d\overrightarrow{\mu} = \Psi dx$ and $d\overrightarrow{\alpha} = \Gamma dx$ we obtain the following direct corollary (see also [5, Theorem 3.3]).

Corollary 3.5. *Let $\Psi \in (L^1(\mathbb{R}^d))^{(t)}$, $\Phi \in (W_0^1)^{(r)}$ satisfy (2.2), and X be a separated Ψ-sampling set for $V^p(\Phi)$. Then there exists $\epsilon_0 > 0$ such that X is a Γ-sampling set for $V^p(\Phi)$, whenever $\Gamma \in (L^1(\mathbb{R}^d))^{(t)}$ and $\|\Psi - \Gamma\|_{(L^1(\mathbb{R}^d))^{(t)}} < \epsilon_0$.*

As a consequence of Theorems 3.1 and 3.4 we obtain the following combined perturbation result and its corollary, which is essentially Theorem 3.4 in [5].

Theorem 3.6. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$. Then there exists $\epsilon_0 > 0$ such that the sampling model*

$(X, \Theta, \overrightarrow{\alpha})$ is also p-stable, whenever $\overrightarrow{\alpha} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, $\Theta \in (W_0^1)^{(r)}$, and $\|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0$.

Corollary 3.7. *Let $\Psi \in (L^1(\mathbb{R}^d))^{(t)}$, $\Phi \in (W_0^1)^{(r)}$ satisfy (2.2), and X be a separated Ψ-sampling set for $V^p(\Phi)$. Then there exists $\epsilon_0 > 0$ such that X is a Γ-sampling set for $V^p(\Theta)$, whenever $\Gamma \in (L^1(\mathbb{R}^d))^{(t)}$, $\Theta \in (W_0^1)^{(r)}$ and $\|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\Psi - \Gamma\|_{(L^1(\mathbb{R}^d))^{(t)}} < \epsilon_0$.*

An error in the location of the sampling points $\{x_j\}$ is what is often called jitter error (see e.g., [6, 7] and the references therein). This error can be modeled as a perturbation of the sampling set X. For this reason, our next perturbation results deal with an altered sampling set $\widetilde{X} = X + \Delta = \{x_j + \delta_j\}_{j \in J}$, where $\Delta = \{\delta_j\}_{j \in J} \subset \mathbb{R}^d$. We use the standard notation for $\|\Delta\|_\infty = \sup\{\|\delta_j\| : j \in J\}$.

Theorem 3.8. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$. Then there exists $\epsilon_0 > 0$ such that the sampling model $(X + \Delta, \Phi, \overrightarrow{\mu})$ is also p-stable, whenever $\|\Delta\|_\infty < \epsilon_0$.*

Remark 3.1. The above theorem is an analog of Theorem 3.6 in [6], where $r = t = 1$, $p = 2$, and $\mu = \mu^1 = \delta_0$.

As a direct corollary of Theorems 3.6 and 3.8 we get the following combined result.

Theorem 3.9. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$. Then there exists $\epsilon_0 > 0$ such that the sampling model $(X+\Delta, \Theta, \overrightarrow{\alpha})$ is also p-stable, whenever $\overrightarrow{\alpha} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, $\Theta \in (W_0^1)^{(r)}$, and $\|\Delta\|_\infty + \|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0$.*

We leave it to the reader to formulate other perturbation theorems resulting from different combinations of Theorems 3.1, 3.4, and 3.8. We conclude this section with a slightly stronger version (due to Lemma 2.2) of Theorem 3.9.

Theorem 3.10. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$ and U be its sampling operator. Let also $(X + \Delta, \Theta, \overrightarrow{\alpha})$ be a perturbed sampling model with the sampling operator U_Δ. Then for every $\epsilon > 0$ there exists $\epsilon_0 > 0$ such that $\|U - U_\Delta\| < \epsilon$, whenever $\overrightarrow{\alpha} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, $\Theta \in (W_0^1)^{(r)}$, and*

$$\|\Delta\|_\infty + \|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0.$$

3.2. **Perfect reconstruction and localized frames.**

In this section we show that a frame algorithm can be used to reconstruct $f \in V^2(\Phi)$ from its samples. We also obtain a useful modification of the above results using the theory of localized frames developed in

[15] (see Definition 3.3). In the previous section, the number $p \in [1,\infty]$ was fixed, that is, we stated, for example, that if X is a $\overrightarrow{\mu}$-sampling set for $V^p(\Phi)$, then X is a $\overrightarrow{\mu}$-sampling set for $V^p(\Theta)$ for *the same* $p \in [1,\infty]$, as soon as Θ is sufficiently close to Φ in the appropriate norm. Here, we claim that if X is a $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$, then X is a $\overrightarrow{\mu}$-sampling set for $V^p(\Theta)$ for *all* $p \in [1,\infty]$, as soon as Θ is sufficiently close to Φ, Φ satisfies a mild decay condition, and $\overrightarrow{\mu}$ belongs to $\mathcal{M}_s(\mathbb{R}^d)$ for some $s > d$. It is natural to ask whether one can replace $V^2(\Phi)$ in the above statement with $V^q(\Phi)$, *for some* $q \in [1,\infty]$. Under certain assumptions the answer is "yes", but it turns out to be a much harder problem as shown in [2].

Definition 3.1. Let $\mathcal{H}$ be a Hilbert space of functions and V a closed subspace of $\mathcal{H}$. Let $\{\Psi_{x_j} = (\psi^1_{x_j}, \dots, \psi^t_{x_j})^T\}_{j\in J}$ be a countable collection of vectors of functions in V. We say that $\{\Psi_{x_j}\}_{j\in J}$ is a frame for V if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|_{\mathcal{H}} \le \|\langle f, \Psi_{x_j}\rangle\|_{(\ell^2(J))^{(t)}} \le B\|f\|_{\mathcal{H}}, \text{ for all } f \in V,$$

where $\langle f, \Psi_{x_j}\rangle = (\langle f, \psi^1_{x_j}\rangle, \dots, \langle f, \psi^t_{x_j}\rangle) \in \mathbb{C}^t$.

Remark 3.2. Notice that the above is not quite the standard definition of a frame in a Hilbert space. This is due to the way we defined the norm in (2). Nevertheless, it is easily seen that $\{\Psi_{x_j}\}_{j\in J}$ is a frame for V according to the above definition if and only if $\{\psi^i_{x_j},\ i = 1, 2, \dots, t,\ j \in J\}$ is a frame for V according to the standard definition. The frame bounds, however, may be different.

Definition 3.2. Let V be a closed subspace of the Hilbert space $\mathcal{H}$. Let $\{\Psi_{x_j} = (\psi^1_{x_j}, \dots, \psi^t_{x_j})^T\}_{j\in J}$ be a frame for V. The frame operator associated with the frame $\{\Psi_{x_j}\}_{j\in J}$ is the operator $S : V \to V$ defined by $S(f) = \sum_{j\in J}\langle f, \Psi_{x_j}\rangle\Psi_{x_j}$, for all $f \in V$. The (canonical) dual frame $\{\widetilde{\Psi}_{x_j}\}_{j\in J}$ of the frame $\{\Psi_{x_j}\}_{j\in J}$ is a sequence of vectors given by $\{\widetilde{\Psi}_{x_j} = (\widetilde{\psi}^1_{x_j}, \dots, \widetilde{\psi}^t_{x_j})^T\}_{j\in J}$, where $\widetilde{\psi}^s_{x_j} = S^{-1}\psi^s_{x_j}$, $1 \le s \le t$.

Remark 3.3. It is well know that a frame operator S is bounded, invertible, self-adjoint, and positive [12]. Hence, the canonical dual frame is well defined. There may exist other dual frames but we will refrain from defining the notion.

The next proposition shows that a frame algorithm can be used to reconstruct a function from its samples.

Proposition 3.11. *Let* $\Phi \in (W^1_0)^{(r)}$, $\overrightarrow{\mu} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, *and* X *be a* $\overrightarrow{\mu}$*-sampling set for* $V^2(\Phi)$. *Then there exists a sequence of vectors of functions* $\{\Psi_{x_j}\}_{j\in J}$, *which is a frame for* $V^2(\Phi)$ *and* $\langle f, \Psi_{x_j}\rangle = (f * \overrightarrow{\mu})(x_j)$

for all $f \in V^2(\Phi)$ and $j \in J$. Moreover, every function $f \in V^2(\Phi)$ can be recovered from the sequence of its samples $\{(f * \overrightarrow{\mu})(x_j)\}_{j\in J}$ via

$$f(x) = \sum_{j\in J} (f * \overrightarrow{\mu})(x_j)\widetilde{\Psi}_{x_j}(x), \tag{3.3}$$

where $\{\widetilde{\Psi}_{x_j}\}_{j\in J}$ is the dual frame of $\{\Psi_{x_j}\}_{j\in J}$ and the series (3.3) converges unconditionally in $V^2(\Phi)$.

The frame $\{\Psi_{x_j}\}_{j\in J}$ constructed in the previous proposition will be called a $(\overrightarrow{\mu}, X)$-*sampling frame* for $V^2(\Phi)$. The main idea of this section is to use the fact that if such a frame is localized then it is also a Banach frame [15] for $V^p(\Phi)$, $p \in [1,\infty)$.

Remark 3.4. Observe that, in general, the frame operator S is the product of the *analysis operator* $T : V \to (\ell^2(J))^{(t)}$, defined by $Tf = \{\langle f, \Psi_{x_j}\rangle\}_{j\in J} = \{(\langle f, \psi^1_{x_j}\rangle, \dots, \langle f, \psi^t_{x_j}\rangle)\}_{j\in J}$ and its adjoint, that is $S = T^*T$. Since Φ generates a Riesz basis, it is immediate that in case of a $(\overrightarrow{\mu}, X)$-sampling frame its analysis operator is isomorphic to the sampling operator $U = U_{(X,\Phi,\overrightarrow{\mu})}$.

Definition 3.3. Let V be a closed subspace of the Hilbert space $\mathcal{H}$. Let $\{\Psi_{x_j} = (\psi^1_{x_j}, \dots, \psi^t_{x_j})^T\}_{j\in J}$ be a frame for V, and $\{G_k = (g^1_k, \dots, g^r_k)^T\}_{k\in\mathbb{Z}^d}$ be a Riesz basis for V, i.e., a condition similar to (2.2) is satisfied. We say that the frame $\{\Psi_{x_j}\}_{j\in J}$ is (polynomially) s-localized with respect to the Riesz basis $\{G_k\}_{k\in\mathbb{Z}^d}$, if

$$|\langle G_k, \Psi^T_{x_j}\rangle| \le C_1(1 + |x_j - k|)^{-s}, \tag{3.4}$$

and

$$|\langle \widetilde{G}_k, \Psi^T_{x_j}\rangle| \le C_2(1 + |x_j - k|)^{-s}, \tag{3.5}$$

for all $j \in J$ and $k \in \mathbb{Z}^d$. Here, the constants $C_1, C_2 > 0$ are independent of j and k, $|\langle G_k, \Psi^T_{x_j}\rangle| = \sum_{i=1}^r \sum_{l=1}^t |\langle g^i_k, \psi^l_{x_j}\rangle|$, $\{\widetilde{G}_k\}_{k\in\mathbb{Z}^d}$ is the dual Riesz basis of $\{G_k\}_{k\in\mathbb{Z}^d}$, and $|\langle \widetilde{G}_k, \Psi^T_{x_j}\rangle|$ is defined similarly to $|\langle G_k, \Psi^T_{x_j}\rangle|$.

Remark 3.5. Let V be a closed subspace of a Hilbert space $\mathcal{H}$. Assume that $\{G_k = (g^1_k, \dots, g^r_k)^T\}_{k\in\mathbb{Z}^d}$ is a Riesz basis for V. The dual Riesz basis of the Riesz basis $\{G_k\}_{k\in\mathbb{Z}^d}$ is the sequence of vectors $\{\widetilde{G}_k = (\widetilde{g}^1_k, \dots, \widetilde{g}^r_k)^T\}_{k\in\mathbb{Z}^d}$ satisfying $\langle \widetilde{G}_k, G^T_l\rangle = \delta_{kl} I$, where I is the $r \times r$ identity matrix, and δ_{kl} is the Kronecker delta. Since a Riesz basis $\{G_k\}$ is also a frame, $\{\widetilde{G}_k\}$ is, in fact, the canonical dual frame for $\{G_k\}$. In this case it is the unique dual frame.

Definition 3.4. Let $\Phi = (\phi^1, \dots, \phi^r)^T \in (W_0^1)^{(r)} \subset (L^2(\mathbb{R}^d))^{(r)}$ and $s > d$. We say that Φ is an s-localized Riesz generator for $V^2(\Phi)$, denoted $\Phi \in \mathcal{W}_s$, if

- $\{\Phi_k = \Phi(\cdot - k)\}_{k \in \mathbb{Z}^d}$ generates a Riesz basis for $V^2(\Phi)$, i.e., condition (2.2) holds for $p = 2$;
- The components of Φ satisfy the decay condition

$$|\phi^i(x)| \le C_0^i (1 + |x|)^{-s}, \tag{3.6}$$

  for all $1 \le i \le r$ and some $C_0^i > 0$ independent of $x \in \mathbb{R}^d$.

Remark 3.6. If $\Phi \in \mathcal{W}_s$, then (2.2) holds for every $p \in [1, \infty]$ as shown in [3].

The following is the main result of subsection 3.2.

Theorem 3.12. *Let $s > d$, $\Phi \in \mathcal{W}_s$, and $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. Assume that X is a $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$, and $\{\Psi_{x_j}\}_{j \in J}$ is the $(\overrightarrow{\mu}, X)$-sampling frame for $V^2(\Phi)$. Then*

- *X is a $\overrightarrow{\mu}$-sampling set for $V^p(\Phi)$ for all $p \in [1, \infty]$.*
- *If $\{\widetilde{\Psi}_{x_j}\}$ is the dual frame for $\{\Psi_{x_j}\}_{j \in J}$, then*

$$f = \sum_{j \in J} (f * \overrightarrow{\mu})(x_j) \widetilde{\Psi}_{x_j}, \text{ for all } f \in V^p(\Phi), \tag{3.7}$$

  where the series converges unconditionally in $V^p(\Phi)$, $p \in [1, \infty)$.

Next, we combine Theorem 3.12 with the perturbation results of the previous section. The proofs are immediate.

Theorem 3.13. *Let $s > d$, $\Phi \in \mathcal{W}_s$, and $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. Assume that X is a separated $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$. Then there exists $\epsilon_0 > 0$ such that for every $\Theta \in \mathcal{W}_s$ satisfying $\|\Phi - \Theta\|_{(W^1)^{(r)}} < \epsilon_0$, there exists a $(\overrightarrow{\mu}, X)$-sampling frame $\{\Psi_{x_j}\}_{j \in J}$ for $V^2(\Theta)$. Moreover,*

- *X is a $\overrightarrow{\mu}$-sampling set for $V^p(\Theta)$ for all $p \in [1, \infty]$.*
- *If $\{\widetilde{\Psi}_{x_j}\}$ is the dual frame for $\{\Psi_{x_j}\}_{j \in J}$, then*

$$f = \sum_{j \in J} (f * \overrightarrow{\mu})(x_j) \widetilde{\Psi}_{x_j}, \text{ for all } f \in V^p(\Theta),$$

  where the series converges unconditionally in $V^p(\Theta)$, $p \in [1, \infty)$.

Theorem 3.14. *Let $s > d$, $\Phi \in \mathcal{W}_s$, and $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. Assume that X is a separated $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$. Then there exists $\epsilon_0 > 0$ such that for every $\overrightarrow{\alpha} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$ satisfying $\|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0$, there exists an $(\overrightarrow{\alpha}, X)$-sampling frame $\{\Psi_{x_j}\}_{j \in J}$ for $V^2(\Phi)$. Moreover,*

- *X is an $\overrightarrow{\alpha}$-sampling set for $V^p(\Phi)$ for all $p \in [1,\infty]$.*
- *If $\{\widetilde{\Psi}_{x_j}\}$ is the dual frame for $\{\Psi_{x_j}\}_{j\in J}$, then*

$$f = \sum_{j\in J} (f * \overrightarrow{\mu})(x_j)\widetilde{\Psi}_{x_j}, \text{ for all } f \in V^p(\Phi),$$

where the series converges unconditionally in $V^p(\Phi)$, $p \in [1,\infty)$.

Theorem 3.15. *Let $s > d$, $\Phi \in \mathcal{W}_s$, and $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. Assume that X is a separated $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$. Then there exists $\epsilon_0 > 0$ such that for every $\Delta = \{\delta_j,\ j \in J\}$ satisfying $\|\Delta\|_\infty < \epsilon_0$ there exists a $(\overrightarrow{\mu}, X + \Delta)$-sampling frame $\{\Psi_{x_j}\}_{j\in J}$ for $V^2(\Phi)$. Moreover,*

- *$X + \Delta$ is a $\overrightarrow{\mu}$-sampling set for $V^p(\Phi)$ for all $p \in [1,\infty]$.*
- *If $\{\widetilde{\Psi}_{x_j}\}$ is the dual frame for $\{\Psi_{x_j}\}_{j\in J}$, then*

$$f = \sum_{j\in J} (f * \overrightarrow{\mu})(x_j + \delta_j)\widetilde{\Psi}_{x_j}, \text{ for all } f \in V^p(\Phi),$$

where the series converges unconditionally in $V^p(\Phi)$, $p \in [1,\infty)$.

Theorem 3.16. *Let $s > d$, $\Phi \in \mathcal{W}_s$, and $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. Assume that X is a separated $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$. Then there exists $\epsilon_0 > 0$ such that for every $\Theta \in \mathcal{W}_s$ and $\overrightarrow{\alpha} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$ satisfying $\|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0$, there exists an $(\overrightarrow{\alpha}, X)$-sampling frame $\{\Psi_{x_j}\}_{j\in J}$ for $V^2(\Theta)$. Moreover,*

- *X is an $\overrightarrow{\alpha}$-sampling set for $V^p(\Theta)$ for all $p \in [1,\infty]$.*
- *If $\{\widetilde{\Psi}_{x_j}\}$ is the dual frame for $\{\Psi_{x_j}\}_{j\in J}$, then*

$$f = \sum_{j\in J} (f * \overrightarrow{\alpha})(x_j)\widetilde{\Psi}_{x_j}, \text{ for all } f \in V^p(\Theta),$$

where the series converges unconditionally in $V^p(\Theta)$, $p \in [1,\infty)$.

Theorem 3.17. *Let $s > d$, $\Phi \in \mathcal{W}_s$, and $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. Assume that X is a separated $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$. Then there exists $\epsilon_0 > 0$ such that for every $\Delta = \{\delta_j,\ j \in J\}$, $\Theta \in \mathcal{W}_s$, and $\overrightarrow{\alpha} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$ satisfying $\|\Delta\|_\infty + \|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0$, there exists an $(\overrightarrow{\alpha}, X + \Delta)$-sampling frame $\{\Psi_{x_j}\}_{j\in J}$ for $V^2(\Theta)$. Moreover,*

- *$X + \Delta$ is an $\overrightarrow{\alpha}$-sampling set for $V^p(\Theta)$ for all $p \in [1,\infty]$.*
- *If $\{\widetilde{\Psi}_{x_j}\}$ is the dual frame for $\{\Psi_{x_j}\}_{j\in J}$, then*

$$f = \sum_{j\in J} (f * \overrightarrow{\alpha})(x_j + \delta_j)\widetilde{\Psi}_{x_j}, \text{ for all } f \in V^p(\Theta),$$

where the series converges unconditionally in $V^p(\Theta)$, $p \in [1,\infty)$.

Remark 3.7. The crucial result for the proof of the theorems in this section is Jaffard's non-commutative extension of the classical Wiener's Tauberian Lemma (see Theorem 5 in [15]). It states that if an invertible matrix has an off-diagonal decay defined by inequalities similar to (3.4) and (3.5), then the inverse matrix has the same off-diagonal decay. There exist other extensions of Wiener's Lemma which deal with different types of off-diagonal decay (see, for example, [10, 16]). Many of those could be used to obtain results similar to, say, Theorem 3.17.

3.3. **Imperfect reconstruction.**

In practice, we know that a perturbation exists because of imperfections of measuring devices, errors, etc. However, we can only estimate this perturbation and may not even know its nature. Here we show that even if we use a model $(X, \Phi, \overrightarrow{\mu})$ for reconstructing a signal from a perturbed model $(\widetilde{X}, \Theta, \overrightarrow{\alpha})$ (or vice versa), the reconstruction error depends continuously on the perturbation in the cases studied above.

As before, let U be the sampling operator for a p-stable sampling model $(X, \Phi, \overrightarrow{\mu})$ and U_Δ be the sampling operator for a perturbed model $(\widetilde{X}, \Theta, \overrightarrow{\alpha})$, where $\widetilde{X} = X + \Delta = \{x_j + \delta_j\}_{j\in J}$. The sampling operator U_Δ can be thought of as a $t \times r$ matrix of operators given by

$$U_\Delta = \begin{pmatrix} U_\Delta^{1,1} & \dots & U_\Delta^{r,1} \\ \vdots & & \vdots \\ U_\Delta^{1,t} & \dots & U_\Delta^{r,t} \end{pmatrix},$$

where for each $1 \le i \le r$ and $1 \le l \le t$ the operator $U_\Delta^{i,l}$ is defined by a bi-infinite matrix with entries $(U_\Delta^{i,l})_{j,k} = (\theta^i * \alpha^l)(x_j + \delta_j - k)$, $j \in J$, $k \in \mathbb{Z}^d$.

We let U^* be an operator defined by the following $r \times t$ matrix of operators from $(\ell^p(J))^{(t)}$ into $(\ell^p(\mathbb{Z}^d))^{(r)}$:

$$U^* = \begin{pmatrix} \overline{U^{1,1}} & \dots & \overline{U^{1,t}} \\ \vdots & & \vdots \\ \overline{U^{r,1}} & \dots & \overline{U^{r,t}} \end{pmatrix},$$

where for each $1 \le i \le r$ and $1 \le l \le t$, the operator $\overline{U^{i,l}}$ is defined by a bi-infinite matrix with entries $(\overline{U^{i,l}})_{j,k} = \overline{(\phi^i * \mu^l)(x_j - k)}$, where $\overline{z}$ denotes the conjugate of the complex number z. The operator $(U_\Delta)^*$ is defined similarly. Notice that this definition implies $\|U^*\|_{p,op} = \|U\|_{p,op}$, and $(U^*)^* = U$. Moreover, if U satisfies (2.6), then U^* satisfies

$$\eta_p \|D\|_{(\ell^p(J))^{(t)}} \le \|U^* D\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \le \beta_p \|D\|_{(\ell^p(J))^{(t)}}, \tag{3.8}$$

for all $D \in (\ell^p(J))^{(t)}$. Observe also that if $p = 2$ then U^* is, indeed, the Hilbert adjoint of U. Hence, if the sampling model $(X, \Phi, \overrightarrow{\mu})$ is 2-stable, U^*U is isomorphic to the frame operator S for the sampling frame $\{\Psi_{x_j}\}$, see Remark 3.4. Therefore, U^*U is invertible and positive. Moreover, the operator $(U^*U)^{-1}U$ is a left inverse for the sampling operator U and it is isomorphic to the synthesis operator used for the reconstruction. Hence, the importance of the following result.

Theorem 3.18. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$. Assume that its sampling operator U satisfies (2.6) and the operator U^*U is invertible. Let $\epsilon \in (0, -\beta_p + \sqrt{\beta_p^2 + \eta_p^2})$ and $(\widetilde{X}, \Theta, \overrightarrow{\alpha})$ be a perturbed sampling model such that its sampling operator U_Δ satisfies $\|U - U_\Delta\| < \epsilon$. Define $\nu = \nu(\epsilon) = \eta_p^{-2}\epsilon(\epsilon + 2\beta_p)$. Then $0 < \nu < 1$, the operator $U_\Delta^* U_\Delta$ is invertible, and*

$$\|(U^*U)^{-1}U^* - (U_\Delta^* U_\Delta)^{-1}U_\Delta^*\| < \frac{1}{\eta_p^2}\left(\epsilon + \frac{\nu(\beta_p + \epsilon)}{1 - \nu}\right).$$

Remark 3.8. Observe that if $p = 2$ we do not need to require invertibility of U^*U. As we mentioned above, it follows automatically.

Remark 3.9. If in Theorem 3.18 we let $r = t = 1$, $p = 2$, and $\mu = \mu^1 = \delta_0$, then we obtain an analog of Theorem 3.3 in [6].

Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$. Assume that its sampling operator U satisfies (2.6) and the operator U^*U is invertible. We define the *reconstruction operator* $R = R_{(X, \Phi, \overrightarrow{\mu})}$: $(\ell^p(J))^{(t)} \to V^p(\Phi)$ by

$$RD = \sum_{k \in \mathbb{Z}^d} [(U^*U)^{-1}U^*D]_k^T \Phi(\cdot - k),$$

$D = (d^1, \dots, d^t)^T$ in $(\ell^p(J))^{(t)}$.

Then as an immediate consequence of Theorems 3.10 and 3.18, we have the following result.

Theorem 3.19. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$. Assume that its sampling operator U is such that U^*U is invertible. Let R be the reconstruction operator. Then for every $\epsilon > 0$ there exists $\epsilon_0 > 0$ such that for every $\Delta = \{\delta_j,\ j \in J\}$, $\Theta \in (W_0^1)^{(r)}$, and $\overrightarrow{\alpha} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$ satisfying*

$$\|\Delta\|_\infty + \|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0,$$

we have

$$\|R((g * \overrightarrow{\alpha})(X + \Delta)) - f\|_{L^p} < \epsilon,\ f = \sum_{k \in \mathbb{Z}^d} C_k^T \Phi_k,\ g = \sum_{k \in \mathbb{Z}^d} C_k^T \Theta_k,$$

for all $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$.

Theorem 3.19 tells us that the reconstruction error is, indeed, controlled in a continuous fashion by each and all of the perturbation errors studied in this paper.

Our final result is a combination of the above theorem with the results of section 3.2.

Theorem 3.20. *Let* $(X, \Phi, \overrightarrow{\mu})$ *be a 2-stable sampling model such that* $\Phi \in \mathcal{W}_s$ *and* $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. *Let* R *be the reconstruction operator for* $(X, \Phi, \overrightarrow{\mu})$. *Then for every* $\epsilon > 0$ *there exists* $\epsilon_0 > 0$ *such that for every* $\Delta = \{\delta_j,\ j \in J\}$, $\Theta \in \mathcal{W}_s$, *and* $\overrightarrow{\alpha} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$ *satisfying*

$$\|\Delta\|_\infty + \|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} < \epsilon_0,$$

we have

$$\|R((g * \overrightarrow{\alpha})(X + \Delta)) - f\|_{L^p} < \epsilon,\ f = \sum_{k \in \mathbb{Z}^d} C_k^T \Phi_k,\ g = \sum_{k \in \mathbb{Z}^d} C_k^T \Theta_k,$$

for all $p \in [1, \infty]$ *and all* $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$.

The proofs in the following section show implicitly how numerical estimates for ϵ_0 in Theorems 3.19 and 3.20 may be obtained.

4. Proofs

4.1. Auxiliary results.

We begin with technical results that are needed for the main proofs.

Lemma 4.1. *Let* $\phi \in W_0^1$, *and* $\mu \in \mathcal{M}(\mathbb{R}^d)$. *Then:*

$$\phi * \mu \in W_0^1, \text{ and} \tag{4.1}$$

$$\|\phi * \mu\|_{W^1} \leq 2^d \|\phi\|_{W^1} \|\mu\|. \tag{4.2}$$

Proof. Note that if $\mu = 0$, the proof is immediate. Assume now $\mu \neq 0$, i.e. $\|\mu\| > 0$. Let $\epsilon > 0$ be given. Since $\phi \in W_0^1$, then ϕ is uniformly continuous in $\mathbb{R}^d$. Therefore, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|\phi(w) - \phi(w_1)| < \frac{\epsilon}{\|\mu\|}, \quad \text{whenever} \quad \|w - w_1\| < \delta. \tag{4.3}$$

Let $z_0 \in \mathbb{R}^d$ be given, and let $z \in \mathbb{R}^d$ be such that $\|z - z_0\| < \delta$. Then we have

$$\begin{aligned} |(\phi * \mu)(z) - (\phi * \mu)(z_0)| &= \left| \int_{\mathbb{R}^d} \phi(z-y) d\mu(y) - \int_{\mathbb{R}^d} \phi(z_0 - y) d\mu(y) \right| \\ &= \left| \int_{\mathbb{R}^d} (\phi(z-y) - \phi(z_0 - y)) d\mu(y) \right| \\ &\leq \int_{\mathbb{R}^d} |\phi(z-y) - \phi(z_0-y)| d|\mu|(y). \end{aligned}$$

Since $\|(z-y)-(z_0-y)\| = \|z-z_0\| < \delta$, for all $y \in \mathbb{R}^d$, then it follows from (4.3) that $\int_{\mathbb{R}^d} |(\phi(z-y) - \phi(z_0-y))| \, d|\mu|(y) < \int_{\mathbb{R}^d} \frac{\epsilon}{\|\mu\|} d|\mu|(y) = \epsilon$. Since z_0 and $\epsilon > 0$ are arbitrary, we obtain the continuity of $\phi * \mu$ in $\mathbb{R}^d$.

Let us show (4.2). Let $\phi \in W^1$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be given. Then

$$\|\phi * \mu\|_{W^1} = \sum_{k \in \mathbb{Z}^d} \operatorname*{esssup}_{x \in [0,1]^d} \left| \int_{\mathbb{R}^d} \phi(x+k-y) \, d\mu(y) \right| \leq$$
$$\sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \operatorname*{esssup}_{x \in [0,1]^d} |\phi(x+k-y)| \, d|\mu|(y) \leq$$
$$\int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}^d} \operatorname*{esssup}_{x \in [0,1]^d} |\phi(x+k-y)| \right) d|\mu|(y) = \int_{\mathbb{R}^d} \|\phi(\cdot - y)\|_{W^1} \, d|\mu|(y).$$

Since $\|\phi(\cdot - y)\|_{W^1} \leq 2^d \|\phi\|_{W^1}$, for all $y \in \mathbb{R}^d$, we get

$$\int_{\mathbb{R}^d} \|\phi(\cdot - y)\|_{W^1} d|\mu|(y) \leq \int_{\mathbb{R}^d} 2^d \|\phi\|_{W^1} d|\mu|(y) = 2^d \|\phi\|_{W^1} \|\mu\|.$$

Therefore, we get (4.2). $\square$

The next proposition collects basic facts about Wiener amalgam spaces, shift invariant spaces $V^p(\Phi)$, and separated sets in $\mathbb{R}^d$.

Proposition 4.2. *Let $\Phi \in (W_0^1)^{(r)}$, $\overrightarrow{\mu} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, $f = \sum_{k \in \mathbb{Z}^d} C_k^T \Phi_k$, where $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$, and $\Phi_k = \Phi(\cdot - k)$, for all $k \in \mathbb{Z}^d$. Let also $X = \{x_j, j \in J\}$ be a separated set in $\mathbb{R}^d$ with a separation constant $\delta > 0$. Then*

$$\Phi * \overrightarrow{\mu} \in (W_0^1)^{(r \times t)}; \tag{4.4}$$

$$\|\Phi * \overrightarrow{\mu}\|_{(W^1)^{(r\times t)}} \leq 2^d \|\Phi\|_{(W^1)^{(r)}} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}; \tag{4.5}$$

$$V^p(\Phi) \subset W_0^p, \text{ for all } 1 \leq p \leq \infty; \tag{4.6}$$

$$\|f\|_{W^p} \leq \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \|\Phi\|_{(W^1)^{(r)}}; \tag{4.7}$$

(4.8) $\|f(X)\|_{\ell^p(J)} \leq \mathcal{N}\|f\|_{W^p}$, *where* $\mathcal{N} = \mathcal{N}(\delta, p, d) = (\frac{\sqrt{d}}{\delta} + 1)^{d/p}$.

Proof. First, Lemma 4.1 immediately implies (4.4).

Next, to prove (4.5) consider $\Phi = (\phi^1, \dots, \phi^r)^T \in (W_0^1)^r$ and $\overrightarrow{\mu} = (\mu^1, \dots, \mu^t) \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$. Then using (4.2), we obtain

$$\begin{aligned} \|\Phi * \overrightarrow{\mu}\|_{(W^1)^{(r\times t)}} &= \sum_{j=1}^{t}\sum_{i=1}^{r}\|\phi^i * \mu^j\|_{W^1} \leq \\ \sum_{j=1}^{t}\sum_{i=1}^{r} 2^d\|\phi^i\|_{W^1}\|\mu^j\| &= 2^d\|\Phi\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}. \end{aligned}$$

Next, we prove (4.7). Consider $1 \leq p < \infty$ and $f = \sum_{k\in\mathbb{Z}^d} C_k^T\Phi_k$. For each $1 \leq s \leq r$ let $a^s(l) = \operatorname*{esssup}_{x\in[0,1]^d}|\phi^s(x+l)|$, for all $l \in \mathbb{Z}^d$. Then $\|a^s\|_{\ell^1(\mathbb{Z}^d)} = \|\phi^s\|_{W^1}$. Consequently, $\|a\|_{(\ell^1(\mathbb{Z}^d))^{(r)}} = \|\Phi\|_{(W^1)^{(r)}}$, where $a = (a^1, \dots, a^r)^T$, and $\Phi = (\phi^1, \dots, \phi^r)^T$. Hence,

$$\operatorname*{esssup}_{x\in[0,1]^d}|f(x+l)| \leq \sum_{s=1}^{r}\sum_{k\in\mathbb{Z}^d}|c^s(k)|\operatorname*{esssup}_{x\in[0,1]^d}|\phi^s(x+l-k)| = \sum_{s=1}^{r}(a^s * |c^s|)(l).$$

By using Young and triangular inequalities, we have

$$\|f\|_{W^p} \leq \sum_{s=1}^{r}\|a^s * |c^s|\|_{\ell^p} \leq \sum_{s=1}^{r}\|a^s\|_{\ell^1}\|c^s\|_{\ell^p}.$$

Consequently, $\|f\|_{W^p} \leq \|C\|_{(\ell^p(\mathbb{Z}^d))^r}\|\Phi\|_{(W^1)^{(r)}}$.

Next, let us show (4.6). Let $f \in V^p(\Phi)$ be given. Then $f = \sum_{k\in\mathbb{Z}^d} C_k^T\Phi_k$, for some $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$. Since (4.7) implies $f \in W^p$, it remains to show the continuity of f. Let us first consider the case $1 \leq p < \infty$. We observe that $W^p \subset W^\infty = L^\infty(\mathbb{R}^d)$ (see Theorem 2.1 in [4]), and, hence,

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq d_1\|f\|_{W^p}, \tag{4.9}$$

for some $d_1 > 0$ independent of f. Let $f_n = \sum_{|k|\leq n} C_k^T\Phi_k$ be a partial sum of f. Since $\Phi \in (W_0^1)^{(r)}$, then $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions, and from (4.7) and (4.9) we obtain

$$\|f - f_n\|_{L^\infty(\mathbb{R}^d)} \leq d_1\|\Phi\|_{(W^1)^{(r)}}\left(\sum_{i=1}^{r}\left(\sum_{|k|>n}|c_k^i|^p\right)^{1/p}\right).$$

Therefore, the sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to the function f. Thus, f is a continuous function as well. To treat the case $p=\infty$, we choose a sequence $\{\Phi_n\}_{n\geq 1}$ of continuous functions with compact support (see Theorem 3.1 in [4] for details) such that $\|\Phi_n-\Phi\|_{(W^1)^{(r)}}\to 0$ as $n\to\infty$. Set $f_n(x)=\sum_{k\in\mathbb{Z}^d} C_k^T\Phi_n(x-k)$. Since the sum is locally finite, then each f_n is continuous. By using (4.7) once again, we estimate

$$\|f_n-f\|_{L^\infty(\mathbb{R}^d)}\leq d_1\|C\|_{(\ell^\infty(\mathbb{Z}^d))^{(r)}}\|\Phi_n-\Phi\|_{(W^1)^{(r)}}.$$

It follows that the sequence of continuous functions $\{f_n\}_{n\geq 1}$ converges uniformly to f. Hence, f is a continuous function as well.

Finally, let us prove (4.8). Since $X=\{x_j, j\in J\}\subset\mathbb{R}$ is separated with a separation constant $\delta>0$, then $\inf_{j\neq k}|x_j-x_k|\geq\delta$. Consequently, there exist at most $([\frac{\sqrt{d}}{\delta}]+1)^d$ sampling points in every d-dimensional hypercube $[0,1]^d+l$, $l\in\mathbb{Z}^d$. Therefore,

$$\sum_{j:x_j\in[0,1]^d+l}|f(x_j)|^p\leq(\delta^{-1}\sqrt{d}+1)^d\operatorname*{esssup}_{x\in[0,1]^d}|f(x)|^p,$$

and, hence, $\|f(X)\|_{\ell^p(J)}\leq\mathcal{N}\|f\|_{W^p}$, for all $f\in W^p$, where $\mathcal{N}=(\delta^{-1}\sqrt{d}+1)^{d/p}$. □

Using (4.4) and (4.5), we obtain the following result.

Corollary 4.3. *Let* $\Lambda:(W_0^1)^{(r)}\times(\mathcal{M}(\mathbb{R}^d))^{(t)}\longrightarrow(W_0^1)^{(r\times t)}$ *be defined by* $\Lambda(\Phi,\overrightarrow{\mu})=\Phi*\overrightarrow{\mu}$*. Then* Λ *is a bounded bilinear form, and* $\|\Lambda\|\leq 2^d$*, where*

$$\|\Lambda\|=\sup\{\|\Lambda(\Phi,\overrightarrow{\mu})\|_{(W_0^1)^{(r\times t)}}:\|\Phi\|_{(W_0^1)^{(r)}}\leq 1,\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}\leq 1\}.$$

The following lemma, proved, for example, in [5], states that a small perturbation of a Riesz basic sequence remains a Riesz basic sequence.

Lemma 4.4. *Let* $\Phi\in(W^1)^{(r)}$ *satisfy (2.2). Then there exists* $\epsilon_0>0$ *such that every* $\Theta\in(W^1)^{(r)}$ *satisfying* $\|\Phi-\Theta\|_{(W^1)^{(r)}}\leq\epsilon<\epsilon_0$*, also satisfies (2.2), for some* $0<m_p'\leq M_p'<\infty$ *and*

$$m_p'\geq m_p-\epsilon\qquad\text{and}\qquad M_p'\leq\|\Phi\|_{(W^1)^{(r)}}+\epsilon. \tag{4.10}$$

4.2. **Proofs for Section 3.1.**

Now we are ready to prove the first of our main results.

Proof of Theorem 3.1.

Proof. Assume that $\overrightarrow{\mu}\in(\mathcal{M}(\mathbb{R}^d))^{(t)}$, $\Phi\in(W_0^1)^{(r)}$ satisfies (2.2), and $X=\{x_j, j\in J\}\subset\mathbb{R}^d$ satisfies (2.5). We want to find $\epsilon_0>0$ such that whenever $\|\Phi-\Theta\|_{(W^1)^{(r)}}\leq\epsilon<\epsilon_0$, then (3.1) takes place for some

$0 < A_p' \le B_p' < \infty$. Assume $0 < \epsilon < m_p$. Then, by Lemma 4.4, $\Theta \in (W^1)^{(r)}$ satisfies (2.2) and we can use representations $g = \sum_{k\in\mathbb{Z}^d} C_k^T \Theta_k$ and $f = \sum_{k\in\mathbb{Z}^d} C_k^T \Phi_k$, $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$. Consequently, we have

$$\begin{aligned}
\frac{1}{M_p'}\|g\|_{L^p} &\le \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \le \frac{1}{m_p}\|\sum_{k\in\mathbb{Z}^d} C_k^T\Phi_k\|_{L^p} = \frac{1}{m_p}\|f\|_{L^p} \\
&\le \frac{A_p^{-1}}{m_p}\|(f * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \\
&= \frac{A_p^{-1}}{m_p}\left\|\left(\left(\sum_{k\in\mathbb{Z}^d} C_k^T\Phi_k\right) * \overrightarrow{\mu}\right)(X)\right\|_{(\ell^p(J))^{(t)}} \\
&= \frac{A_p^{-1}}{m_p}\sum_{l=1}^{t}\left\|\left(\left(\sum_{k\in\mathbb{Z}^d} C_k^T\Phi_k\right) * \mu^l\right)(X)\right\|_{\ell^p(J)} \\
&\le \frac{A_p^{-1}}{m_p}\sum_{l=1}^{t}\left\|\left(\sum_{k\in\mathbb{Z}^d} C_k^T\Xi_k^l\right)(X)\right\|_{\ell^p(J)} \\
&+ \frac{A_p^{-1}}{m_p}\|(g * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}},
\end{aligned}$$

where

$$\Xi_k^l := ((\phi_k^1 - \theta_k^1) * \mu^l, \ldots, (\phi_k^r - \theta_k^r) * \mu^l), \quad l = 1, \ldots, t. \tag{4.11}$$

Since Φ and Θ are elements of $(W_0^1)^{(r)}$ and $\overrightarrow{\mu} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, then by (4.4), we have $\Xi^l = (\Phi - \Theta) * \mu^l \in (W_0^1)^{(r)}$, for $l = 1, \ldots, t$. Hence, using (4.5), (4.6) and condition (2.2) for $g = \sum_{k\in\mathbb{Z}^d} C_k^T\Theta_k$, we have

$$\begin{gathered}
\textstyle\sum_{l=1}^{t}\left\|(\sum_{k\in\mathbb{Z}^d} C_k^T\Xi_k^l)(X)\right\|_{\ell^p(J)} \le \\
2^d\mathcal{N}\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}\|\Phi-\Theta\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} \le \\
\tfrac{2^d\mathcal{N}\|\Phi-\Theta\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p'}\|g\|_{L^p}.
\end{gathered}$$

Therefore,

$$\begin{aligned}
\frac{1}{M_p'}\|g\|_{L^p} &\le \frac{A_p^{-1}2^d\mathcal{N}\|\Phi-\Theta\|_{(\mathcal{W}^1)^{(r)}}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p m_p'}\|g\|_{L^p} + \\
&+ \frac{A_p^{-1}}{m_p}\|(g * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}}.
\end{aligned}$$

Hence,

$$\left(\frac{A_p m_p}{M_p'} - \frac{2^d \mathcal{N} \|\Phi - \Theta\|_{(W^1)^{(r)}} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p'}\right) \|g\|_{L^p} \tag{4.12}$$
$$\leq \|(g * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}}.$$

On the other hand, since $\Theta \in (W_0^1)^{(r)}$ and $\overrightarrow{\mu} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$, it follows from (4.4) that $(\theta^1 * \mu^l, \dots, \theta^r * \mu^l) \in (W_0^1)^{(r)}$, $l = 1, \dots, t$. Therefore, (4.7), (4.8) and the first of the estimates in (4.10) imply that

$$\begin{aligned}
\|(g * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} &= \|((\sum_{k \in \mathbb{Z}^d} C_k^T \Theta_k) * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \\
&\leq \mathcal{N} \|((\sum_{k \in \mathbb{Z}^d} C_k^T \Theta_k) * \overrightarrow{\mu})\|_{(W^p)^{(r)}} \\
&\leq 2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} \|\sum_{k \in \mathbb{Z}^d} C_k^T \Theta_k\|_{(W^p)^{(r)}} \\
&\leq 2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \|\Theta\|_{(W^1)^{(r)}} \\
&\leq \frac{2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p'} (\|\Phi\|_{(W^1)^{(r)}} + \epsilon) \|g\|_{L^p} \\
&\leq \frac{2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} (\|\Phi\|_{(W^1)^{(r)}} + \epsilon)}{m_p - \epsilon} \|g\|_{L^p}.
\end{aligned}$$

Hence,

$$\|(g * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \leq \left(\frac{2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} (\|\Phi\|_{(W^1)^{(r)}} + \epsilon)}{m_p - \epsilon}\right) \|g\|_{L^p}. \tag{4.13}$$

Using the estimates (4.10) and the left hand side of the inequality (4.12), we can obtain an explicit upper bound ϵ_0 for ϵ from

$$\frac{A_p m_p}{\|\Phi\|_{(W^1)^{(r)}} + \epsilon} - \frac{2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p - \epsilon} \epsilon = 0.$$

This is equivalent to the quadratic equation

$$\epsilon^2 + C_p \epsilon - \frac{A_p m_p^2}{2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}} = 0,$$

where

$$C_p = \|\Phi\|_{(W^1)^{(r)}} + \frac{A_p m_p}{2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}.$$

Let ϵ_0 be the positive solution of the previous equation, i.e.,

$$\epsilon_0 = \frac{1}{2}\left(\sqrt{C_p^2 + \frac{4A_p m_p^2}{2^d\mathcal{N}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}} - C_p\right).$$

Then, for $0<\epsilon<\epsilon_0<m_p$, we use (4.12), (4.13), and (4.10) to obtain

$$A_p^{'} = \frac{A_p m_p}{\|\Phi\|_{(W^1)^{(r)}}+\epsilon} - \frac{2^d\mathcal{N}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p-\epsilon}\epsilon,$$

$$B_p^{'} = \frac{2^d\mathcal{N}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}(\|\Phi\|_{(W^1)^{(r)}}+\epsilon)}{m_p-\epsilon},$$

and the proof is complete. □

Proof of Theorem 3.4.

Proof. Let $f=\sum_{k\in\mathbb{Z}^d} C_k^T\Phi_k \in V^p(\Phi)$, $C\in(\ell^p(\mathbb{Z}^d))^{(r)}$. We have

$$\begin{aligned}
A_p\|f\|_{L^p} &\leq \|(f*\overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}}\\
&\leq \|(f*(\overrightarrow{\mu}-\overrightarrow{\alpha}))(X)\|_{(\ell^p(J))^{(t)}} + \|(f*\overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}}\\
&= \sum_{l=1}^{t}\|(f*(\mu^l-\alpha^l))(X)\|_{\ell^p(J)} + \|(f*\overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}}\\
&= \sum_{l=1}^{t}\|((\sum_{k\in\mathbb{Z}^d} C_k^T\Phi_k)*(\mu^l-\alpha^l))(X)\|_{\ell^p(J)} + \|(f*\overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}}.
\end{aligned}$$

Since $\overrightarrow{\mu}$ and $\overrightarrow{\alpha}$ are in $(\mathcal{M}(\mathbb{R}^d))^{(t)}$, and $\Phi\in(W_0^1)^{(r)}$, then Proposition 4.2 implies $\Omega^l = (\phi^1*(\mu^l-\alpha^l),\ldots,\phi^r*(\mu^l-\alpha^l)) \in (W_0^1)^{(r)}$, for $l=1,\ldots,t$. Using Proposition 4.2 once again we have:

$$\begin{aligned}
A_p\|f\|_{L^p} &\leq \sum_{l=1}^{t}\mathcal{N}\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}\|\Omega^l\|_{(W^1)^{(r)}} + \|(f*\overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}}\\
&\leq 2^d\mathcal{N}\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}\|\Phi\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}-\overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} + \|(f*\overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}}.
\end{aligned}$$

Taking into account $\Phi\in(W^1)^{(r)}$ also satisfies (2.2), and f satisfies (2.5), then it follows

$$2^d\mathcal{N}\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}\|\Phi\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}-\overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} \leq \frac{2^d\mathcal{N}\|f\|_{L^p}\|\Phi\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}-\overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p}.$$

Hence,

(4.14)
$$\left(A_p - \frac{2^d\mathcal{N}\|\Phi\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}-\overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p}\right)\|f\|_{L^p} \leq \|(f*\overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}}.$$

On the other hand, since $f \in V^p(\Phi)$ satisfies (2.5), we have

$$\begin{aligned}
\|(f*\overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}} &\leq \|(f*(\overrightarrow{\alpha}-\overrightarrow{\mu}))(X)\|_{(\ell^p(J))^{(t)}} + \|(f*\overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \\
&= \sum_{l=1}^{t} \|(f*(\alpha^l-\mu^l))(X)\|_{\ell^p(J)} + \|(f*\overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \\
&\leq \sum_{l=1}^{t} \|((\sum_{k\in\mathbb{Z}^d} C_k^T\Phi_k)*(\alpha^l-\mu^l))(X)\|_{\ell^p(J)} + B_p\|f\|_{L^p} \\
&\leq 2^d\mathcal{N}\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}\|\Phi\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}-\overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} + B_p\|f\|_{L^p}.
\end{aligned}$$

Using condition (2.2), we obtain:

(4.15)

$$\|(f*\overrightarrow{\alpha})(X)\|_{(\ell^p(J))^{(t)}} \leq \left(\frac{2^d\mathcal{N}\|\Phi\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}-\overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p} + B_p\right)\|f\|_{L^p}.$$

From (4.14) and (4.15), by choosing

$$\epsilon_0 = \frac{A_p m_p}{2^d\mathcal{N}\|\Phi\|_{(W^1)^{(r)}}},$$

we obtain for $0<\epsilon<\epsilon_0$,

$$A_p^{'} = A_p - \frac{2^d\mathcal{N}\|\Phi\|_{(W^1)^{(r)}}}{m_p}\epsilon, \quad \text{and}$$

$$B_p^{'} = B_p + \frac{2^d\mathcal{N}\|\Phi\|_{(W^1)^{(r)}}}{m_p}\epsilon.$$

□

Proof of theorem 3.6.

The conclusion of the theorem is essentially obvious at this point. We proceed with a formal proof in order to obtain estimates for ϵ_0 and the bounds $A_p^{'}$ and $B_p^{'}$ of X as an $\overrightarrow{\alpha}$-sampling set for $V^p(\Theta)$.

Proof. Let $0<\epsilon_1<\frac{1}{2}\left(\sqrt{C_p^2+\frac{4A_pm_p^2}{2^d\mathcal{N}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}}-C_p\right)$, where

$$C_p = \|\Phi\|_{(W^1)^{(r)}} + \frac{A_p m_p}{2^d\mathcal{N}\|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}.$$

Then, by Theorem 3.1, X is a $\overrightarrow{\mu}$-sampling set for $V^p(\Theta)$ as soon as

$$\|\Phi-\Theta\|_{(W^1)^{(r)}} \leq \epsilon_1.$$

Moreover,

$$A_p^{''}\|g\|_{L^p} \leq \|(g*\overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \leq B_p^{''}\|g\|_{L^p}, \text{ for all } g\in V^p(\Theta),$$

where

$$A_p'' = \frac{A_p m_p}{\|\Phi\|_{(W^1)^{(r)}} + \epsilon_1} - \frac{2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}}{m_p - \epsilon_1}\epsilon_1$$

and

$$B_p'' = \frac{2^d \mathcal{N} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} (\|\Phi\|_{(W^1)^{(r)}} + \epsilon_1)}{m_p - \epsilon_1}.$$

Assume now that

$$0 < \epsilon_2 \leq \frac{A_p''(m_p - \epsilon_1)}{2^d \mathcal{N}(\|\Phi\|_{(W^1)^{(r)}} + \epsilon_1)}.$$

Then, by Theorem 3.4, X is an $\overrightarrow{\alpha}$-sampling set for $V^p(\Theta)$ as soon as

$$\|\Phi - \Theta\|_{(W^1)^{(r)}} \leq \epsilon_1 \text{ and } \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} \leq \epsilon_2.$$

Hence, if $0 < \epsilon < \epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, we obtain the sampling bounds

$$A_p' = A_p'' - \frac{2^d \mathcal{N}(\|\Phi\|_{(W^1)^{(r)}} + \epsilon_1)}{m_p - \epsilon_1}\epsilon_2,$$

and

$$B_p' = B_p'' + \frac{2^d \mathcal{N}(\|\Phi\|_{(W^1)^{(r)}} + \epsilon_1)}{m_p - \epsilon_1}\epsilon_2,$$

as soon as

$$\|\Phi - \Theta\|_{(W^1)^{(r)}} + \|\overrightarrow{\mu} - \overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}} \leq \epsilon < \epsilon_0.$$

□

Proof of Theorem 3.8.

The theorem is immediately implied by Lemma 2.2 and the following result.

Lemma 4.5. *Let $(X, \Phi, \overrightarrow{\mu})$ be a p-stable sampling model for some $p \in [1, \infty]$ and $\widetilde{X} = X + \Delta$. Let U be the sampling operator for $(X, \Phi, \overrightarrow{\mu})$ and U_Δ be the sampling operator for $(\widetilde{X}, \Phi, \overrightarrow{\mu})$. Then $\|U - U_\Delta\| \to 0$ as $\|\Delta\|_\infty \to 0$.*

Proof. We recall that for any $\gamma > 0$, the function $\operatorname{osc}_\gamma g$ on $\mathbb{R}^d$ is defined by

$$\operatorname{osc}_\gamma g(x) = \sup_{|\Delta x| < \gamma} |g(x + \Delta x) - g(x)|.$$

From Lemma 8.1 in [4] it follows that if $g \in W_0^1$, then $\operatorname{osc}_\gamma g \in W^1$, and $\|\operatorname{osc}_\gamma g\|_{W^1} \to 0$ as $\gamma \to 0$. Therefore, by applying Proposition 4.2 we get

$$\operatorname{osc}_\gamma \Phi * \overrightarrow{\mu} \in (W^1)^{(r \times t)}, \text{ and } \|\operatorname{osc}_\gamma \Phi * \overrightarrow{\mu}\|_{(W^1)^{(r \times t)}} \to 0 \text{ as } \gamma \to 0,$$

where

$$\operatorname{osc}_\gamma \Phi * \overrightarrow{\mu} = \begin{pmatrix} \operatorname{osc}_\gamma \phi^1 * \mu^1 & \dots & \operatorname{osc}_\gamma \phi^1 * \mu^t \\ \vdots & & \vdots \\ \operatorname{osc}_\gamma \phi^r * \mu^1 & \dots & \operatorname{osc}_\gamma \phi^r * \mu^t \end{pmatrix}.$$

For any $m \in \mathbb{Z}^d$ there exist at most $([\delta^{-1}\sqrt{d}]+1)^d$ sampling points in every hypercube $[0,1]^d + m$. We set $X_m = X \bigcap([0,1]^d + m)$, $m \in \mathbb{Z}^d$, and, for each $1 \le i \le r$ and $1 \le l \le t$, define the sequence

$$b^{i,l}(m) := \operatorname{esssup}_{x\in[0,1]^d} \{\operatorname{osc}_{\|\Delta\|_\infty} (\phi^i * \mu^l)(x+m)\}, \quad m \in \mathbb{Z}^d.$$

Then $\|b^{i,l}\|_{\ell^1(\mathbb{Z}^d)} = \|\operatorname{osc}_{\|\Delta\|_\infty} (\phi^i * \mu^l)\|_{W^1}$ and, hence,

$$\|b\|_{(\ell^1(\mathbb{Z}^d))^{(r\times t)}} = \|\operatorname{osc}_{\|\Delta\|_\infty} \Phi * \overrightarrow{\mu}\|_{(W^1)^{(r\times t)}}.$$

For $1 \le i \le r$ and $1 \le l \le t$ we have

$$\begin{aligned} \|(U^{i,l} - U^{i,l}_\Delta)c^i\|^p_{\ell^p(J)} &= \sum_{x_j\in X} \left| \sum_{k\in\mathbb{Z}^d} c^i_k \left((\phi^i * \mu^l)(x_j - k)) - (\phi^i * \mu^l)(x_j + \delta_j - k)\right)\right|^p \\ &\le \sum_{x_j\in X} \left(\sum_{k\in\mathbb{Z}^d} |c^i_k| \operatorname{osc}_{\|\Delta\|_\infty} (\phi^i * \mu^l)(x_j - k)\right)^p \\ &\le \sum_{m\in\mathbb{Z}^d} \mathcal{N}^p \left(\sum_{k\in\mathbb{Z}^d} |c^i_k| b^{i,l}(m-k)\right)^p \\ &= \mathcal{N}^p \||c^i| * b^{i,l}\|^p_{\ell^p(\mathbb{Z}^d)}, \end{aligned}$$

where $\mathcal{N} = (\delta^{-1}\sqrt{d}+1)^{d/p}$. By using Young's inequality we obtain

$$\begin{aligned} \mathcal{N}^p \||c^i| * b^{i,l}\|^p_{\ell^p(\mathbb{Z}^d)} &\le \mathcal{N}^p \|c^i\|^p_{\ell^p(\mathbb{Z}^d)} \|b^{i,l}\|^p_{l^1} \\ &= \mathcal{N}^p \|c^i\|^p_{\ell^p(\mathbb{Z}^d)} \|\operatorname{osc}_{\|\Delta\|_\infty} \phi^i * \mu^l\|^p_{W^1}. \end{aligned}$$

Consequently,

$$\|U^{i,l} - U^{i,l}_\Delta\| \le \mathcal{N} \|\operatorname{osc}_{\|\Delta\|_\infty} \phi^i * \mu^l\|_{W^1}.$$

Hence,

$$\|U - U_\Delta\| \le \mathcal{N} \|\operatorname{osc}_{\|\Delta\|_\infty} \Phi * \overrightarrow{\mu}\|_{(W^1)^{(r\times t)}} \to 0 \quad \text{as} \quad \|\Delta\|_\infty \to 0,$$

and the lemma is proved. □

Proof of Theorem 3.10.

Proof. The proof of Theorem 3.10 is hidden in the proofs of Theorems 3.1, 3.4, and 3.8. In particular, keeping the notation of the proof of Theorem 3.1, we have

$$\|((f-g) * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \le 2^d \mathcal{N} \|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \|\Phi - \Theta\|_{(W^1)^{(r)}} \|\overrightarrow{\mu}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}.$$

Hence, Theorem 3.10 is true, when $\overrightarrow{\mu} = \overrightarrow{\alpha}$ and $X = X + \Delta$. Keeping the notation of the proof of Theorem 3.4, we have

$$\|(f*(\overrightarrow{\alpha}-\overrightarrow{\mu}))(X)\|_{(\ell^p(J))^{(t)}} \leq 2^d \mathcal{N}\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}\|\Phi\|_{(W^1)^{(r)}}\|\overrightarrow{\mu}-\overrightarrow{\alpha}\|_{(\mathcal{M}(\mathbb{R}^d))^{(t)}}.$$

This inequality implies Theorem 3.10 when $\Phi = \Theta$ and $X = X + \Delta$. Combining these results with Theorem 3.8 via the standard $\epsilon/3$ argument we prove the general case. $\square$

4.3. **Proofs for Section 3.2.**

We begin with an auxiliary technical result for the convolution of functions with measures.

Lemma 4.6. *Let $\Phi = (\phi^1, \dots, \phi^r)^T$ be a vector of continuous functions, $s > d$, and $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. If $|\phi^i(x)| \leq C_0^i(1 + |x|)^{-s}$ for all $1 \leq i \leq r$, then*

$$|(\Phi * \overrightarrow{\mu})(x)| \leq C_1(1 + |x|)^{-s};$$

the constants $C_0^i > 0$, $1 \leq i \leq r$, and $C_1 > 0$ are independent of $x \in \mathbb{R}^d$.

Proof. For $1 \leq i \leq r$ and $1 \leq j \leq t$ we have

$$\begin{aligned} |(\phi^i * \mu^j)(x)| &\leq \int_{\mathbb{R}^d} |\phi^i(x - y)| d|\mu^j|(y) \\ &\leq C_0^i \int_{\mathbb{R}^d} (1 + |x - y|)^{-s} d|\mu^j|(y). \end{aligned}$$

Since $(1 + |u + w|)^{-l} \leq (1 + |u|)^l (1 + |w|)^{-l}$, for all $u, w \in \mathbb{R}^d$, and $l \geq 0$, we have

$$\begin{aligned} |(\phi^i * \mu^j)(x)| &\leq C_0^i \int_{\mathbb{R}^d} (1 + |y|)^s (1 + |x|)^{-s} d|\mu^j|(y) \\ &= C_0^i (1 + |x|)^{-s} \int_{\mathbb{R}^d} (1 + |y|)^s d|\mu^j|(y) \\ &\leq C_1^{i,j} (1 + |x|)^{-s}, \end{aligned}$$

where the last inequality follows from $\mu^j \in \mathcal{M}_s(\mathbb{R}^d)$. Therefore,

$$|(\Phi * \overrightarrow{\mu})(x)| \leq C_1(1 + |x|)^{-s},$$

where $C_1 = \sum_{i=1}^r \sum_{j=1}^t C_1^{i,j}$. $\square$

Remark 4.1. If $\{\Phi_k\}_{k \in \mathbb{Z}^d}$ is an s-localized Riesz generator for $V^2(\Phi)$, as in Definition 3.4, then, by Lemma 14(a) in [15], we have that $\{\widetilde{\Phi}_k\}_{k \in \mathbb{Z}^d}$ is also an s-localized Riesz generator for $V^2(\Phi)$. Consequently, by Lemma 4.6 we have

$$|(\widetilde{\Phi} * \overrightarrow{\mu})(x)| \leq D_1(1 + |x|)^{-s}, \tag{4.16}$$

for some $D_1 > 0$ independent of $x \in \mathbb{R}^d$.

Proof of Proposition 3.11.

Proof. Let X be a $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$, $\overrightarrow{\mu} \in (\mathcal{M}(\mathbb{R}^d))^{(t)}$. Then, by definition, there exist constants $0 < A_2 \le B_2 < \infty$ such that

$$A_2\|f\|_{L^2} \le \|(f*\overrightarrow{\mu})(X)\|_{(\ell^2(J))^{(t)}} \le B_2\|f\|_{L^2}, \text{for all } f \in V^2(\Phi). \tag{4.17}$$

Fix $x_j \in X$. Then, for each $1 \le i \le t$, the function $g^i_{x_j}\colon V^2(\Phi) \to \mathbb{C}$ given by $g^i_{x_j}(f) = (f * \mu^i)(x_j)$ is a bounded linear functional on the closed subspace $V^2(\Phi)$ of $L^2(\mathbb{R}^d)$ because $|g^i_{x_j}(f)| \le B_2\|f\|_{L^2}$ for all $f \in V^2(\Phi)$. Consequently, by Riesz representation theorem, there exists $\psi^i_{x_j} \in V^2(\Phi)$ such that $g^i_{x_j}(f) = \langle f, \psi^i_{x_j}\rangle$ for all $f \in V^2(\Phi)$. It follows immediately from (4.17) and Definition 3.1 that $\Psi_{x_j} = (\psi^1_{x_j}, \dots, \psi^t_{x_j})^T$ is a frame for $V^2(\Phi)$. Hence, every $f \in V^2(\Phi)$ can be recovered via $f = \sum_{j\in J}\langle f, \Psi_{x_j}\rangle\widetilde{\Psi}_{x_j}$, where $\{\widetilde{\Psi}_{x_j} = (\widetilde{\psi}^1_{x_j}, \dots, \widetilde{\psi}^t_{x_j})^T\}_{j\in J}$ is a dual frame of $\{\Psi_{x_j}\}_{j\in J}$ and the series converges unconditionally in $V^2(\Phi)$. Since $\langle f, \Psi_{x_j}\rangle = (f * \overrightarrow{\mu})(x_j)$ for all $j \in J$, we get (3.3). ☐

Next, we show that if the generator Φ and the measures $\overrightarrow{\mu}$ satisfy an appropriate decay condition then the $(\overrightarrow{\mu}, X)$-sampling frame $\{\Psi_{x_j}\}$ obtained above is s-localized.

Proposition 4.7. *Let $s > d$, $\Phi \in \mathcal{W}_s$, and $\overrightarrow{\mu} \in (\mathcal{M}_s(\mathbb{R}^d))^{(t)}$. If X is a $\overrightarrow{\mu}$-sampling set for $V^2(\Phi)$, then the $(\overrightarrow{\mu}, X)$-sampling frame $\{\Psi_{x_j}\}$ is s-localized with respect to the Riesz basis $\{\Phi_k\}_{k\in\mathbb{Z}^d}$.*

Proof. Since $\{\Phi_k\}_{k\in\mathbb{Z}^d}$ is an s-localized Riesz generator for $V^2(\Phi)$, the components of Φ satisfy (3.6), and Lemma 4.6 implies

$$|\langle\Phi_k, \Psi^T_{x_j}\rangle| = |(\Phi * \overrightarrow{\mu})(x_j - k)| \le C_1(1 + |x_j - k|)^{-s},$$

for some $C_1 > 0$ independent of $j \in J$ and $k \in \mathbb{Z}^d$. On the other hand, it follows from Remark 4.1 that the dual Riesz basis $\{\widetilde{\Phi}_k\}_{k\in\mathbb{Z}^d}$ is also an s-localized Riesz generator for $V^2(\Phi)$, and its components also satisfy (3.6). Therefore, using Lemma 4.6 once again, we get

$$|\langle\widetilde{\Phi}_k, \Psi^T_{x_j}\rangle| = |(\widetilde{\Phi} * \overrightarrow{\mu})(x_j - k)| \le D_1(1 + |x_j - k|)^{-s},$$

for some $D_1 > 0$ independent of $j \in J$ and $k \in \mathbb{Z}^d$. Hence, $\{\Psi_{x_j}\}$ satisfies all conditions of Definition 3.3. ☐

We conclude this subsection with the proof of the main result of section 3.2.

Proof of Theorem 3.12

Proof. Assume the hypotheses of Theorem 3.12. By Propositions 3.11 and 4.7, there exists a $(\overrightarrow{\mu}, X)$-sampling frame $\{\Psi_{x_j}\}_{j\in J}$ for $V^2(\Phi)$, which is s-localized with respect to the Riesz basis $\{\Phi_k\}_{k\in\mathbb{Z}^d}$ and satisfies

$$\langle f, \Psi_{x_j}\rangle = (f * \overrightarrow{\mu})(x_j), \text{ for all } f \in V^2(\Phi).$$

Moreover,

$$f = \sum_{j\in J} (f * \overrightarrow{\mu})(x_j)\widetilde{\Psi}_{x_j}, \text{ for all } f \in V^2(\Phi).$$

Consequently, applying Theorem 10(c) in [15], we get

$$f = \sum_{j\in J} (f * \overrightarrow{\mu})(x_j)\widetilde{\Psi}_{x_j}, \text{ for all } f \in V^p(\Phi),$$

where the series converges unconditionally in $V^p(\Phi)$, $1 \le p < \infty$. Moreover, since $\{\Psi_{x_j}\}_{j\in J}$ is an s-localized frame with respect to the Riesz basis $\{\Phi_k\}_{k\in\mathbb{Z}^d}$, then Theorem 10(d) in [15] implies that for each $1 \le p \le \infty$ there exist $0 < A_p \le B_p < \infty$ such that

$$A_p\|f\|_{L^p} \le \|(f * \overrightarrow{\mu})(X)\|_{(\ell^p(J))^{(t)}} \le B_p\|f\|_{L^p}, \text{ for all } f \in V^p(\Phi),$$

i.e., X is a $\overrightarrow{\mu}$-sampling set for $V^p(\Phi)$ and the theorem is proved. □

4.4. **Proofs for section 3.3.**

For the proof of Theorem 3.18 we need the following two lemmas.

Lemma 4.8. *Let the assumptions of Theorem 3.18 hold. Then*

$$\|U^*U - U^*_\Delta U_\Delta\| < \epsilon\,(2\beta_p + \epsilon)\,.$$

Proof. Since $\|U\| = \|U^*\|$ and $\|U - U_\Delta\| = \|U^* - U^*_\Delta\|$,

$$\begin{aligned}
\|U^*U - U^*_\Delta U_\Delta\| &= \|U^*U - U^*U_\Delta + U^*U_\Delta - U^*_\Delta U_\Delta\| \\
&= \|U^*(U - U_\Delta) + (U^* - U^*_\Delta)U_\Delta\| \\
&\le \|U^*\|\|U - U_\Delta\| + \|U^* - U^*_\Delta\|\|U_\Delta\| \\
&\le \|U - U_\Delta\|\,(\|U\| + \|U_\Delta\|) \\
&\le \|U - U_\Delta\|\,(2\|U\| + \|U - U_\Delta\|) \\
&\le \epsilon\,(2\beta_p + \epsilon)\,,
\end{aligned}$$

and the lemma is proved. □

Lemma 4.9. *Let the assumptions of Theorem 3.18 hold. Then* $0 < \nu < 1$, $(U^*_\Delta U_\Delta)^{-1}$ *exists, and* $\|(U^*U)^{-1} - (U^*_\Delta U_\Delta)^{-1}\| < \frac{\nu}{\eta_p^2(1-\nu)}$.

Proof. Since $(U^*U)^{-1}$ exists,

$$U_\Delta^* U_\Delta = U^*U\left(I + (U^*U)^{-1}\left(U_\Delta^* U_\Delta - U^*U\right)\right). \tag{4.18}$$

From (2.6) and (3.8) we get that for all $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$

$$\frac{1}{\beta_p^2}\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \le \|(U^*U)^{-1}C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}} \le \frac{1}{\eta_p^2}\|C\|_{(\ell^p(\mathbb{Z}^d))^{(r)}}.$$

From the above inequalities and Lemma 4.8 we have

$$\begin{gathered}\|(U^*U)^{-1}\left(U_\Delta^* U_\Delta - U^*U\right)\| \le \|(U^*U)^{-1}\|\|U_\Delta^* U_\Delta - U^*U\| \\ \le \tfrac{1}{\eta_p^2}\epsilon\left(2\beta_p + \epsilon\right) < \tfrac{1}{\eta_p^2}(-\beta_p + \sqrt{\beta_p^2 + \eta_p^2})\left(2\beta_p - \beta_p + \sqrt{\beta_p^2 + \eta_p^2}\right) = 1.\end{gathered}$$

Hence, $\nu = \frac{1}{\eta_p^2}\epsilon\left(2\beta_p + \epsilon\right) \in (0,1)$. To simplify the notation, we define

$$M := U^*U, \quad M_\Delta := U_\Delta^* U_\Delta, \text{ and } N := (U^*U)^{-1}\left(U_\Delta^* U_\Delta - U^*U\right).$$

Since $\|N\| \le \nu < 1$, then $(I+N)^{-1}$ exists and is given by the Neumann series

$$(I+N)^{-1} = \sum_{q=0}^{\infty}(-1)^q N^q.$$

From (4.18) we obtain

$$M_\Delta^{-1} = [M(I+N)]^{-1} = (I+N)^{-1}M^{-1}. \tag{4.19}$$

Therefore, $M_\Delta^{-1} = (U_\Delta^* U_\Delta)^{-1}$ exists.

Now we need to give an upper bound for $\|M^{-1} - M_\Delta^{-1}\|$. Using (4.19) we obtain

$$M^{-1} - M_\Delta^{-1} = N(I+N)^{-1}M^{-1}.$$

Consequently,

$$\begin{aligned}\|M^{-1} - M_\Delta^{-1}\| &\le \|N\|\|(I+N)^{-1}\|\|M^{-1}\| \\ &\le \frac{\|N\|}{1-\|N\|}\|M^{-1}\| \le \frac{\nu}{1-\nu}\eta_p^{-2},\end{aligned} \tag{4.20}$$

and the lemma is proved. □

Proof of theorem 3.18.

Proof. Using the notations from Lemmas 4.8, 4.9, and the previous proofs, we get

$$\begin{aligned}\|(U^*U)^{-1}U^* - (U_\Delta^* U_\Delta)^{-1}U_\Delta^*\| &= \|M^{-1}U^* - M_\Delta^{-1}U_\Delta^*\| \\ &= \|M^{-1}U^* - M^{-1}U_\Delta^* + M^{-1}U_\Delta^* - M_\Delta^{-1}U_\Delta^*\| \\ &= \|M^{-1}(U^* - U_\Delta^*) + (M^{-1} - M_\Delta^{-1})U_\Delta^*\| \\ &\le \|M^{-1}\|\|U^* - U_\Delta^*\| + \|M^{-1} - M_\Delta^{-1}\|\|U_\Delta^*\| \\ &\le \frac{1}{\eta_p^2}\left(\epsilon + \frac{\nu\,(\epsilon+\beta_p)}{1-\nu}\right).\end{aligned}$$

□

Proof of Theorem 3.19.

Proof. Let U_Δ be the sampling operator for a perturbed sampling model $(X+\Delta, \Theta, \overrightarrow{\alpha})$. Let also $C \in (\ell^p(\mathbb{Z}^d))^{(r)}$, $f = \sum_{k\in\mathbb{Z}^d} C_k^T \Phi_k$, and $g = \sum_{k\in\mathbb{Z}^d} C_k^T \Theta_k$. Then

$$\|R(g * \overrightarrow{\alpha})(X+\Delta) - f\|_{L^p} \le M_p \left\|((U^*U)^{-1}U^*U_\Delta C - C)\right\|_{\ell^p}.$$

It remains to apply Theorem 3.18 to finish the proof. □

Proof of Theorem 3.20.

Proof. Assume the hypotheses of Theorem 3.20. From Theorem 3.12 we know that, in this case, the sampling model $(X, \Phi, \overrightarrow{\mu})$ is p-stable for every $p \in [1,\infty]$. Hence, in view of Theorem 3.19, the only thing that we need to prove is that the operator U^*U is invertible for all $p \in [1,\infty]$ and not just for $p = 2$.

Taking into account that for each $1 \le i \le r$ and $1 \le l \le t$ the entries of the matrix of the operator $U^{i,l}$ satisfy

$$|(U^{i,l})_{j,k}| = |(\phi^i * \mu^l)(x_j - k)| \le C_1(1 + |x_j - k|)^{-s},$$

for some $C_1 > 0$ independent of $j \in J$ and $k \in \mathbb{Z}^d$, it follows from Lemma 3 in [15] that the matrix of U defines a bounded linear operator from $(\ell^p(\mathbb{Z}^d))^{(r)} \to (\ell^p(J))^{(t)}$ for all $1 \le p \le \infty$. Hence, U^* is also well defined as a bounded linear operator from $(\ell^p(J))^{(t)} \to (\ell^p(\mathbb{Z}^d))^{(r)}$, and, therefore, $U^*U : (\ell^p(\mathbb{Z}^d))^{(r)} \to (\ell^p(\mathbb{Z}^d))^{(r)}$ is a well defined and bounded operator for all $1 \le p \le \infty$. On the other hand, since the operator U^*U is invertible on $(\ell^2(\mathbb{Z}^d))^{(r)}$ and its components $(M^{i,l})_{j,k}$, $1 \le i \le r$, $1 \le l \le r$, satisfy a decay condition

$$|(M^{i,l})_{j,k}| \le C_2(1 + |x_j - k|)^{-s},$$

for some $C_2 > 0$ independent of $j \in J$ and $k \in \mathbb{Z}^d$, then Jaffard's Lemma (see Theorem 5 in [15]) implies that $(U^*U)^{-1} : (\ell^2(\mathbb{Z}^d))^{(r)} \to$

$(\ell^2(\mathbb{Z}^d))^{(r)}$ is also a bounded linear operator defined by a matrix satisfying the same off-diagonal decay condition as U^*U. Consequently, using Lemma 3 in [15] once again, we get that the matrix of $(U^*U)^{-1}$ defines a bounded linear operator on $(\ell^p(\mathbb{Z}^d))^{(r)}$ for all $1 \leq p \leq \infty$. The theorem is proved. □

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